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ON THE GEODESICS AND GEODESIC CIRCLES ON A DEVELOPABLE SURFACE.*

By William Caspar Graustein.

1. If a geodesic† on a developable surface is given parametrically, in terms of its arc s, by $x = x(s)$,‡ the developable is its rectifying developable and is given, in terms of s and a second parameter, η, by

$$(1) \qquad y = x + \eta\left(\frac{R}{T}\alpha - \gamma\right).$$

The treatment in this paper depends upon this means of representing a developable surface.§ If the developable is so given, equations mapping it isometrically upon a plane are easily found. Thus the geodesics and geodesic circles on the developable are determined and formulæ and theorems for them obtained. Algebraic geodesics and geodesic circles are discussed, the intrinsic equations of the geodesic circles on cylinders and cones are set up and the cylinders on which there exist geodesic circles of constant curvature are determined.

The treatment holds for every type of developable, in that (1) is based, not on the developable, but on a geodesic on it. Furthermore, since this geodesic is an arbitrary one on the developable, a general theorem, proved for it, holds for every other one.

2. The ratio of the curvature to the torsion of a geodesic of (1) equals the tangent of the angle under which it intersects the ruling; for if Φ $(0 \leq \Phi < \pi)$ is this angle for the " x-curve," then

$$\tan \Phi = \frac{T}{R}.$$

The direction of the ruling, in terms of Φ, is $\alpha \cos \Phi - \gamma \sin \Phi$, and that of the normal to the surface, suitably oriented, is β.

If η is a function of s, (1) represents the general curve (not a ruling) on the surface. The formulæ for this curve, on the introduction of the symbols, ξ, A, B, as follows:

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* Read before the American Mathematical Society, January 2, 1915.

† The rulings of the developable are not considered as geodesics.

‡ In this paper x_1, x_2, x_3 are the coördinates of the point x; $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$ is the parametric representation of the curve $x = x(t)$; α_1, α_2, α_3, β_1, β_2, β_3, γ_1, γ_2, γ_3 are the direction cosines of the directions α, β, γ of the tangent, principal normal and binormal; R and T are the radii of curvature and torsion.

§ This representation was used by Enneper, "Zur Theorie der Curven doppelter Krümmung," Mathematische Annalen, Bd. 19 (1882), pp. 72–78, to determine the natural equation of the geodesics on a developable which has as edge of regression a given type of curve.

$$s + \eta \cot \Phi = \xi, \qquad \xi'\eta'' - \xi''\eta' = A,$$

$$1 + \eta \frac{d}{ds}(\cot \Phi) = \xi' - \eta' \cot \Phi = B,$$

where the primes denote differentiation with respect to s, are

$$\begin{aligned} ds_y &= \sqrt{\xi'^2 + \eta'^2}\,ds, \\ \sqrt{y'\,|\,y'}\,\alpha_y &= \xi'\alpha - \eta'\gamma, \\ \sqrt{y'\,|\,y'}\sqrt{y'y''\,|\,y'y''}\,\beta_y &= -A\eta'\alpha + \frac{B}{R}(\xi'^2 + \eta'^2)\beta - A\xi'\gamma, \\ \sqrt{y'y''\,|\,y'y''}\,\gamma_y &= \frac{B}{R}\eta'\alpha + A\beta + \frac{B}{R}\xi'\gamma, \end{aligned} \tag{2}$$

where

$$(y'\,|\,y') = \xi'^2 + \eta'^2, \qquad (y'y''\,|\,y'y'') = A^2 + \frac{B^2}{R^2}(\xi'^2 + \eta'^2).$$ [*]

Finally, the geodesic curvature of the curve is

$$\frac{1}{r_y} = \frac{(\gamma_y\,|\,\beta)}{R_y} = \frac{A}{(\xi'^2 + \eta'^2)^{\frac{3}{2}}}. \tag{3}$$

The edge of regression of the developable (if it has one) is given by $B = 0$. For the real points of one sheet of a real developable $B > 0$, and for those of the other sheet, $B < 0$. The x-curve lies wholly on one sheet (if $\eta = 0$, then $B = 1$); hence so does every geodesic.

3. By (2), the curves $\eta = \text{const}$ and the x-curve, $\eta = 0$, have parallel tangents and principal normals in corresponding points, that is, in points on the same ruling. Hence the curves $\eta = \text{const}$ are geodesics on the developable, such that when the developable is applied to the plane they become parallel lines. Accordingly, they shall be called a family of parallel geodesics. Their orthogonal trajectories form a second family of parallel geodesics, given, as is evident from (2), by $\xi = s + \eta \cot \Phi = \text{const}$. Now if the orthogonal system of geodesics, $\xi = \text{const}$, $\eta = \text{const}$, is made parametric, with ξ, η as the parameters, the linear element of the surface becomes $ds^2 = d\xi^2 + d\eta^2$. Hence the equations

$$\xi = s + \eta \cot \Phi, \qquad \eta = \eta \tag{4}$$

define an isometric map of the developable (1) on the (ξ, η) plane.

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[*] And

$$(y'|y') = y_1'^2 + y_2'^2 + y_3'^2,$$

$$(y'y''|y'y'') = (y_2'y_3'' - y_3'y_2'')^2 + (y_3'y_1'' - y_1'y_3'')^2 + (y_1'y_2'' - y_2'y_1'')^2.$$

4. If a geodesic on a developable is considered as a line of shortest distance and a geodesic circle as the locus of a point moving (on the surface) at a fixed geodesic distance from a given point of the surface, their finite equations are, according to the map (4),

$$c_1 + c_2\eta + c_3\xi = 0, \tag{5}$$

$$(\xi - \xi_0)^2 + (\eta - \eta_0)^2 = r^2. \tag{6}$$

The geodesics may, however, be considered as the curves of zero geodesic curvature, or as the curves with principal normals normal to the surface. Either of these points of view leads simply to the differential equation, $A = \xi'\eta'' - \xi''\eta' = 0$, whose integral may be put in the form (5). It will be better to use, instead of (5),

$$\eta = \tan\varphi(\xi - k), \tag{7}$$

though in so doing the geodesics $\eta = \text{const}$ are ruled out. Evidently φ is the angle, suitably measured, between the geodesic and the x-curve, and k determines its point of intersection with the x-curve. Thus the curves $\varphi = \text{const}$ are parallel geodesics cutting the geodesics $\eta = \text{const}$ under the (constant) angle φ; the curves $k = \text{const}$ are the geodesics passing through the fixed point $s = k$ of the x-curve.

For a developable surface geodesic circles may be defined as above or as curves of non-vanishing constant geodesic curvature.* Thus, from (3), their differential equation is

$$r(\xi'\eta'' - \xi''\eta') = (\xi'^2 + \eta'^2)^{\frac{3}{2}}. \tag{8}$$

If for ξ' is set $\eta'\cot\varphi$, and then again, for η', $\xi'\tan\varphi$, where φ is now considered as a function of s, there results, first,

$$\xi' = \varphi' r\cos\varphi, \qquad \eta' = \varphi' r\sin\varphi, \tag{9}$$

and hence the two equations, equivalent to (6):

$$\xi - \xi_0 = r\sin\varphi, \qquad \eta - \eta_0 = -r\cos\varphi. \tag{10}$$

5. The general geodesic w (not one of the family $\eta = \text{const}$) is

$$w = x + \frac{s-k}{\cot\varphi - \cot\Phi}(\alpha\cot\Phi - \gamma).$$

The formulæ (2) become, for this geodesic,

$$\begin{gathered} ds_w = \eta'\csc\varphi ds, \\ \alpha_w = \alpha\cos\varphi - \gamma\sin\varphi, \qquad \beta_w = \beta, \qquad \gamma_w = \alpha\sin\varphi + \gamma\cos\varphi. \end{gathered} \tag{11}$$

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* The two definitions lead to the same curves only for surfaces of constant total curvature.

By application of the Frenet-Serret formulæ R_w and T_w are found:

$$\frac{1}{R_w} = \frac{\sin^2(\Phi - \varphi)}{\sin^2 \Phi} \frac{1}{BR}, \qquad \frac{1}{T_w} = \frac{\sin 2(\Phi - \varphi)}{\sin 2\Phi} \frac{1}{BT}. \tag{12}$$

If Φ_w, where $\cot \Phi_w = R_w/T_w$, is introduced, evidently

$$\cot \Phi_w = \cot(\Phi - \varphi), \qquad \text{or} \qquad \Phi_w = \Phi - \varphi, \qquad -\pi < \Phi_w < \pi.$$

Then the formulæ (12) for the curvature and torsion become

$$R_w \sin^2 \Phi_w = BR \sin^2 \Phi, \qquad T_w \sin 2\Phi_w = BT \sin 2\Phi.$$

From the first formula Euler's equation for normal curvatures in a parabolic point may be reduced. A similar equation for the torsions of the geodesics in a point follows from the second formula.

By means of the formulæ (12) it is easy to establish the following results.*

The product of the curvatures of two orthogonal geodesics in a point equals the square of the torsion of either in the point.

As a geodesic rotates about a point, the rate of change of its curvature with respect to the angle of rotation is twice the torsion of the geodesic in the point:

$$\frac{\partial}{\partial \varphi}\left(\frac{1}{R_w}\right) = -\frac{2}{T_w}. \tag{13}$$

If the radius of curvature, R, of the x-curve (lying wholly on the sheet $B > 0$ of the surface) is taken as never negative, the curvature of an arbitrary curve of the surface in a point of the sheet $B > 0$ is never negative, and, in a point of the sheet $B < 0$, never positive.

6. The ratio of the curvature of a geodesic to its torsion has been shown equal to the tangent of the angle under which the geodesic cuts the ruling. Geometric interpretations may be obtained also for the derivative and the logarithmic derivative of this ratio, by introducing the edge of regression of the developable,

$$X = x - \frac{\alpha \cot \Phi - \gamma}{\dfrac{d}{ds}(\cot \Phi)}. \tag{14}$$

If a point p trace the geodesic with unit velocity and P is the point on the edge of regression corresponding to p, that is, lying on the same ruling, the rate of change of the ratio in p equals the negative reciprocal of the distance of P from the osculating plane in p, and its logarithmic rate of change equals the reciprocal of the distance of P from the normal plane in p.

Now, from (14), the developable is, (a) a cylinder, if and only if the ratio in question is constant,* and (*b*) a cone, if and only if the derivative of the ratio with respect to the arc of the geodesic is constant, not zero.† The geometric interpretation of the ratio and its derivative may be introduced here to obtain geometric criteria for cylinders and cones.‡

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* The first of these is true, for a general surface, in any parabolic point; the second, in any non-singular point.

7. The general geodesic circle z on the developable (1) is

$$z = x + \eta(\alpha \cot \Phi - \gamma),$$

where s and η are connected by (6), or better, are given, by (10), in terms of φ. Now φ, for a point of the geodesic circle z, measures the angle under which it meets the curve $\eta = \text{const}$ through the point, just as it did for the geodesic w; for, by (9), the α_y of (2) becomes $\alpha_w = \alpha \cos \varphi - \gamma \sin \varphi$. But for the geodesics φ was an arbitrary constant, and here it is an auxiliary variable. And since φ entered into (7) only in $\cot \varphi$, there it could be restricted to the interval $0 \leq \varphi < \pi$; on the other hand (10) contains $\sin \varphi$, $\cos \varphi$, so that here φ can at most be restricted to the interval $0 \leq \varphi < 2\pi$. Thus through a given point in a given (unoriented) direction on the developable there pass two geodesic circles of given geodesic curvature.

From (9) and (2), by means of (8) and (12),

$$(z'|z') = \xi'^2 + \eta'^2 = r^2\varphi'^2 = \frac{BR_w}{R}, \qquad rA = (z'|z')^{\frac{3}{2}}.$$

By means of these formulæ (2), in conjunction with (11), gives

$$\frac{1}{R_z^{\,2}} = \frac{1}{R_w^{\,2}} + \frac{1}{r^2}, \tag{15}$$

$$\alpha_z = \alpha_w, \qquad \frac{\beta_z}{R_z} = \frac{\beta_w}{R_w} - \frac{\gamma_w}{r}, \qquad \frac{\gamma_z}{R_z} = \frac{\beta_w}{r} + \frac{\gamma_w}{R_w}.$$

Thus the curvature and the directions of the trihedral of the geodesic circle z in a point are expressed in terms of the corresponding quantities for the geodesic w, tangent to it in the point.§ The first relation says that the difference of the squares of the curvature and geodesic curvature of a geodesic circle in a point depends only on its direction there. Further, the two geodesic circles in the same direction in a point with the same geodesic curvature have the same curvature; their principal normals are equally inclined, on opposite sides, to the normal of the surface.

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* This was first shown by de Saint-Venant, "Sur les lignes courbes non planes," Journal de l'ecole polytechnique, vol. 30 (1845).

† This was first proved by Enneper, "Bemerkungen über Curven doppelter Krümmung," Göttingener Nachrichten, 1886, pp. 134–140.

‡ One of the two for the cone is given by Pirondini, "Sulle linee a doppia curvatura," Giornale di Matematiche di Battaglini, vol. 26 (1888), p. 105 ff.

§ These relations hold for an arbitrary curve on the general surface as the "z-curve" and the normal section tangent to it as the "w-curve" and may be obtained directly from Meusnier's theorem and the definition of geodesic curvature.

8. The torsion of the geodesic circle z plus the derivative, with respect to its arc, of the angle ψ_z, which its osculating plane makes with the tangent plane to the surface, equals the torsion of the tangent geodesic w:

$$\frac{1}{T_z} + \frac{d\psi_z}{ds_z} = \frac{1}{T_w}, \qquad \text{where} \qquad \cos\psi_z = (\gamma_z|\beta_w) = \frac{R_z}{r}.^* \tag{16}$$

Hence

$$\frac{1}{T_z} = \frac{1}{T_w} - \frac{R_z R_w}{r}\frac{d}{ds_z}\left(\frac{1}{R_z}\right).$$

Differentiation of (15) yields

$$\frac{R_z}{R_w}\frac{d}{ds_z}\left(\frac{1}{R_z}\right) = \frac{d}{ds_w}\left(\frac{1}{R_w}\right) + \frac{d\varphi}{ds_z}\frac{\partial}{\partial\varphi}\left(\frac{1}{R_w}\right) = \frac{d}{ds_w}\left(\frac{1}{R_w}\right) - \frac{2}{rT_w},$$

where (13) and the relation $r d\varphi = ds_z$ are used to obtain the final form. Thus the torsion of the geodesic circle is

$$\frac{1}{T_z} = \frac{1}{T_w}\left(1 + 2\frac{R_z^2}{r^2}\right) - \frac{R_z^2}{r^2}\frac{d}{ds_w}\left(\frac{r}{R_w}\right). \tag{17}$$

If φ is increased by π, the last term on the right changes sign, the first term does not. Thus the two geodesic circles in the same direction in a point with the same geodesic curvature have the same torsion only if their common tangent geodesic behaves in the point like a curve of constant curvature, and the same torsion except for sign only if it behaves there like a plane curve. Any two such geodesic circles on a circular cylinder have the same torsion in the point of contact. Finally, inspection shows that the torsions of two geodesic circles in two oriented directions making equal angles with the ruling are equal except for sign.

9. For a particular type of developable given, it is possible to build up from the intrinsic equation of its geodesics and the relation (16), or (17), the natural equation of its geodesic circles, by the use of (15), (12) and the formula $s_z = r\varphi$. Thus for the circular cylinder of radius a, the intrinsic equations of its individual geodesic, $T \sin 2\Phi = 2a$, $R \sin^2 \Phi = a$, and the relation (17) yield the intrinsic equations of its geodesic circles,

$$\frac{1}{T_z} - \left(1 + 2\frac{R_z^2}{r^2}\right)\frac{\sin 2\left(\frac{s_z}{r} - \Phi\right)}{2a} = 0,$$

and the intrinsic equations of its individual geodesic circle,

$$\frac{1}{R_z^2} = \frac{1}{r^2} + \frac{\sin^4 t}{a^2}, \qquad \frac{1}{T_z} = \frac{r^2 \sin^4 t + 3a^2}{r^2 \sin^4 t + a^2}\,\frac{\sin 2t}{2a}, \qquad t = \Phi - \frac{s_z}{r}.$$

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[*] This is Bonnet's formula for the geodesic torsion of a curve on an arbitrary surface. Cf. Eisenhart, Differential Geometry, p. 138.

The intrinsic equation of the geodesic circles on an arbitrary cylinder, for which the intrinsic equation of the geodesics is $\cot \Phi = \text{const}$, is obtained similarly as

$$\frac{1}{T_z} + \frac{d\psi_z}{ds_z} + \frac{\sin \psi_z}{R_z} \cot \left(\frac{s_z}{r} - \Phi \right) = 0.$$

Evidently, if the curvature of a geodesic circle is constant, its torsion is proportional to the cotangent of the ratio of its arc to its radius of geodesic curvature. Further, it may be shown, that there exists one (and hence ∞^1) such geodesic circles on the cylinder only if the intrinsic equation of the directrix may be put in the form, $\bar{R}\, r \tan \psi_z = r^2 - \bar{s}^2$,* where ψ_z is constant.

The intrinsic equation of the geodesic circles on a cone, whose geodesics are given by $ds - c\, d(\cot \Phi) = 0$, turns out to be

$$\frac{dK_z}{ds_z} + \frac{\sin K_z \sin \left(K_z + \dfrac{s_z}{r} \right)}{c + \eta_0 - r \cos \dfrac{s_z}{r}} + \frac{1}{r} = 0, \qquad \text{where} \qquad \cot K_z = \frac{\dfrac{1}{T_z} + \dfrac{d\psi_z}{ds_z}}{\dfrac{\sin \psi_z}{R_z}}.$$

In fact, this determines the geodesics or the geodesic circles according as r is finite or becomes infinite, and applies to the cone when c is finite, to the cylinder when c becomes infinite.

10. A curve is algebraic if it can be expressed parametrically by algebraic functions, and is algebraically rectifiable if it can be so expressed in terms of its arc.

It follows, from (1), that a developable is algebraic, if a single geodesic on it is algebraic; and from (7), that if two non-parallel geodesics are algebraic or if a single geodesic is algebraically rectifiable, every geodesic is not only algebraic but also algebraically rectifiable.*

Inspection of (6) shows further: (*a*) if a geodesic and a geodesic circle are algebraic, every geodesic and every geodesic circle is algebraic; (*b*) if the geodesics are algebraic, the geodesic circles are algebraic—but not algebraically rectifiable, since s_z is algebraic or not with φ ($s_z = r\varphi$), and the trigonometric functions of φ are algebraic.

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* Here $\bar{R}$ is a non-definite quadratic polynomial in $\bar{s}$; the curves for which $\bar{R}$ is a definite quadratic polynomial in $\bar{s}$ are the alysoids (e. g., the catenary) of Cesàro, Nouvelles Annales de Mathématiques, 1886, p. 75.

* Cf. Stäckel, "Über algebraisch rectificierbare Raumkurven," Mathematische Annalen, Bd. 43 (1893), p. 171 ff., where this last result is found as an application of a theory of algebraically rectifiable space curves, developed largely by means of the properties of involutes and evolutes.